FORMULAS

OF

TRIGONOMETRY

SEAVER

Wm. Rotch Ware
May. 1874.

THE FORMULAS

OF

PLANE AND SPHERICAL

TRIGONOMETRY.

COLLECTED AND ARRANGED

FOR THE USE OF STUDENTS AND COMPUTERS.

BY

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Boston and Cambridge:

SEVER, FRANCIS, & CO.

1871.

UNIVERSITY PRESS: WELCH, BIGELOW, & CO.,
CAMBRIDGE.

PREFACE.

I HAVE endeavored to collect and arrange in a manner well adapted to practical use all the trigonometric formulas a student is likely to need in reading other books on mathematics and physics, or in ordinary computations.

At the same time I have adopted a logical order of arrangement, so that the book may be used as a syllabus by teachers who prefer the method of oral instruction. The book contains all the material necessary to a complete treatment of Circular Functions.

As a book of reference, this collection may be found useful in the recitation-room, whenever it seems well to relieve the pupil from the burden of remembering the numerous formulas he may need in his work. It may be used, too, as a book of problems.

Great care has been bestowed on the printing, in the hope that the book might be free from error. I shall be obliged by notice of any which may have escaped detection.

E. P. S.

CAMBRIDGE, 1871.

CONTENTS.

TRIGONOMETRY.

I. Definitions.

1. Two right triangles are *similar*, if an oblique angle of the one is equal to an oblique angle of the other; they are also similar, if any two sides of the one are proportional to the homologous sides of the other. Hence, if one of the oblique angles of a right triangle is given, the other angle is determined, and likewise the values of the six ratios between the three sides; or if any one of these six ratios is given, the values of the other ratios, and the magnitudes of the angles are determined. Thus it is evident that a variation in the magnitude of either of the oblique angles produces a variation in the value of each one of the six ratios. Therefore the ratios of the sides may be regarded as *functions* of either of the oblique angles.

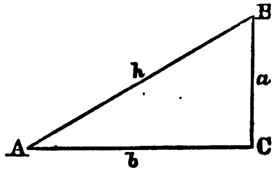
These ratios are called *Trigonometric* or *Circular Functions;* and each has received a distinctive name.

2. The *sine* of an angle in a right triangle $= \dfrac{\text{opposite leg}}{\text{hypotenuse}}$.

The *tangent* of an angle in a right triangle $= \dfrac{\text{opposite leg}}{\text{adjacent leg}}$.

The *secant* of an angle in a right triangle $= \dfrac{\text{hypotenuse}}{\text{adjacent leg}}$.

The *cosine, cotangent,* and *cosecant* of an angle are the sine, tangent, and secant of its *complement;* and since the oblique angles of a right triangle are complements of each other, the cosine, cotangent, and cosecant of one are the sine, tangent, and secant of the other.



3. Let h denote the hypotenuse of a right triangle, a the perpendicular, b the base, A and B the angles opposite a and b respectively. Then

$$\sin A = \frac{a}{h} = \cos B$$

$$\cos A = \frac{b}{h} = \sin B$$

$$\tan A = \frac{a}{b} = \cot B$$

$$\cot A = \frac{b}{a} = \tan B$$

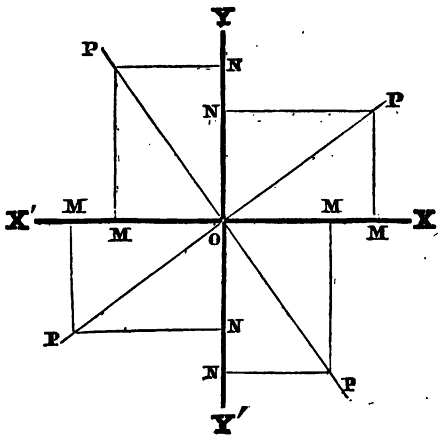
$$\sec A = \frac{h}{b} = \csc B$$

$$\csc A = \frac{h}{a} = \sec B$$

It is evident the *sine* and *cosine* are always *less than unity*, and the *secant* and *cosecant* always *greater*, while the *tangent* and *cotangent* may be *greater or less*, one being greater whenever the other is less.

4. The trigonometric functions may be extended to angles of any magnitude by using *opposite signs* to denote *opposite directions.*

Let X′OX and Y′OY be two lines, called *axes*, which intersect at right angles, and from any point P in the plane drop perpendiculars PM and PN to these axes. Then let



$x = \text{OM} = \text{NP}$,
$y = \text{ON} = \text{MP}$,
$r = \text{OP}$,
$\varphi =$ angle XOP;

and let the positive sign belong to x if it proceeds from O towards the *right*, to y if it proceeds from O upwards, to φ if it represents rotation from OX opposite to that of the hands of a clock, and to r in all its positions. Then for all positions of the point P, $\sin\varphi = \frac{y}{r}$, $\cos\varphi = \frac{x}{r}$, $\tan\varphi = \frac{y}{x}$, $\cot\varphi = \frac{x}{y}$, $\sec\varphi = \frac{r}{x}$, $\csc\varphi = \frac{r}{y}$.

If then P is in the *first quadrant*, or $90° > \varphi > 0°$, *all the functions of* φ *are positive*, since x and y are positive.

If P is in the *second quadrant*, or $180° > \varphi > 90°$, x is negative and y positive, so that the *sine* and *cosecant* of φ are *positive*, and *the other functions negative.*

If P is in the *third quadrant*, or $270° > \varphi > 180°$, x and y are both negative, so that the *tangent* and *cotangent* of φ are *positive*, and *the other functions negative.*

If P is in the *fourth quadrant*, or $360° > \varphi > 270°$, x is positive, and y negative, so that the *cosine* and *secant* of φ are *positive*, and *the other functions negative.*

For every position of P, φ may be measured either positively or negatively, giving two values such that $\varphi'' = \varphi' - 360°$, but the functions of these two values of φ are respectively equal.

In general, 360° may be added to or subtracted from φ without changing the position of P or the values of the functions of φ. Hence, the functions of $(\varphi \pm k\,360°)$ are respectively equal to those of φ.

5. *In a circle whose radius is unity, certain lines may be drawn whose lengths, for a given arc (or angle), have the same numerical values as the ratios above given.* Thus,

The *sine* of an arc is the perpendicular distance of the end of the arc from the diameter passing through the beginning.

The *cosine* is the distance from the centre of the circle to the foot of the sine.

The *tangent* is that part of the geometric tangent drawn at the beginning of the arc, which is intercepted between the point of tangency and the diameter passing through the end of the arc.

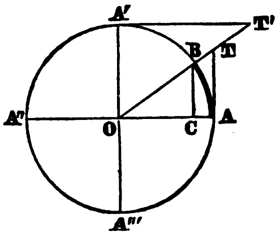
The *cotangent* is that part of the geometric tangent drawn at a quadrant's distance from the beginning of the arc, which is intercepted between the point of tangency and the diameter passing through the end of the arc.

The *secant* is the distance from the centre of the circle to the end of the tangent.

The *cosecant* is the distance from the centre of the circle to the end of the cotangent.

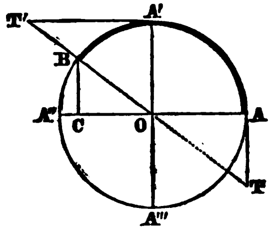
The *versed sine* is the distance from the foot of the sine to the beginning of the arc.

These lines are drawn for an arc terminating in each quadrant, in the figures on the next page. In each figure let the arc φ be represented by the darker portion of the circumference; then $\sin\varphi = CB$, $\cos\varphi = OC$, $\tan\varphi = AT$, $\cot\varphi = A'T'$, $\sec\varphi = OT$, $\csc\varphi = OT'$, $\text{ver}\sin\varphi = CA$.



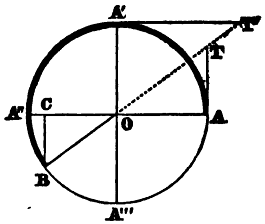
FIRST QUADRANT.

sin +	cot +
cos +	sec +
tan +	csc +



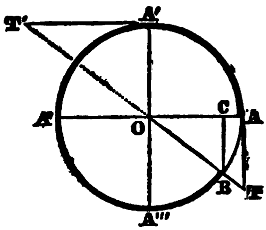
SECOND QUADRANT.

sin +	cot —
cos —	sec —
tan —	csc +



THIRD QUADRANT.

sin —	cot +
cos —	sec —
tan +	csc —



FOURTH QUADRANT.

sin —	cot —
cos +	sec +
tan —	csc —

A *positive* arc is measured from A around in the direction A′, A″, A‴; a *negative* arc, from A in the opposite direction.

It will be seen that both the positive and the negative arc ending at B have the same *sine, cosine,* &c.

II. General Formulas.

Fundamental Relations.

6. $$\begin{cases} \dfrac{1}{\sin\alpha} = \csc\alpha & \dfrac{1}{\csc\alpha} = \sin\alpha \\ \dfrac{1}{\cos\alpha} = \sec\alpha & \dfrac{1}{\sec\alpha} = \cos\alpha \\ \dfrac{1}{\tan\alpha} = \cot\alpha & \dfrac{1}{\cot\alpha} = \tan\alpha \end{cases}$$

7. $$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \sin\alpha\sec\alpha = \frac{\sec\alpha}{\csc\alpha} = \frac{1}{\cos\alpha\csc\alpha}$$

8. $$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \cos\alpha\csc\alpha = \frac{\csc\alpha}{\sec\alpha} = \frac{1}{\sin\alpha\sec\alpha}$$

9. $$\sin^2\alpha + \cos^2\alpha = 1$$

10. $$\sec^2\alpha = 1 + \tan^2\alpha$$

11. $$\csc^2\alpha = 1 + \cot^2\alpha$$

12. $$\text{ver sin}\,\alpha = 1 - \cos\alpha$$

13. $$\sin\alpha = \tfrac{1}{2}\,\text{chord}\,2\alpha$$

14. $$\text{chord}\,\alpha = 2\sin\frac{\alpha}{2}$$

15. *Some Particular Values.*

Angle	Arc	sin	cos	tan	cot	sec	csc
0°	0	0	1	0	∞	1	∞
30°	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\sqrt{\frac{1}{3}}$	$\sqrt{3}$	$2\sqrt{\frac{1}{3}}$	2
45°	$\frac{1}{4}\pi$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\sqrt{\frac{1}{3}}$	2	$2\sqrt{\frac{1}{3}}$
90°	$\frac{1}{2}\pi$	1	0	∞	0	∞	1
120°	$\frac{2}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	-2	$2\sqrt{\frac{1}{3}}$
135°	$\frac{3}{4}\pi$	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5}{6}\pi$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{3}$	$-2\sqrt{\frac{1}{3}}$	2
180°	π	0	-1	0	∞	-1	∞

16. *Cardinal Values.*

Angle	Arc	sin	cos	tan	cot	sec	csc
0°	0	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$
I. qu.		p. i.	p. d.	p. i.	p. d.	p. i.	p. d.
90°	$\frac{1}{2}\pi$	$+1$	± 0	$\pm\infty$	± 0	$\pm\infty$	$+1$
II. qu.		p. d.	n. d.	n. i.	n. d.	n. i.	p. i.
180°	π	± 0	-1	∓ 0	$\mp\infty$	-1	$\pm\infty$
III. qu.		n. d.	n. i.	p. i.	p. d.	n. d.	n. i.
270°	$\frac{3}{2}\pi$	-1	∓ 0	$\pm\infty$	± 0	$\mp\infty$	-1
IV. qu.		n. i.	p. i.	n. i.	n. d.	p. d.	n. d.
360°	2π	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$

NOTE.—In the above table *p.* and *n.* mean *positive* and *negative*; *i.* and *d.* mean *increasing* and *decreasing* in *algebraic* value. For example, the cotangent of an angle in the third quadrant is positive and decreasing with the increase of the angle.

Functions of the Sum and Difference of Arcs or Angles.

17. $$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

18. $$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

19. $$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

20. $$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

21. $$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}=\frac{\cot\beta\pm\cot\alpha}{\cot\beta\cot\alpha\mp1}$$

22. $$\cot(\alpha\pm\beta)=\frac{\cot\beta\cot\alpha\mp1}{\cot\beta\pm\cot\alpha}=\frac{1\mp\tan\alpha\tan\beta}{\tan\alpha\pm\tan\beta}$$

23.
$$\sin(\alpha+\beta+\gamma)=-\sin\alpha\sin\beta\sin\gamma+\sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\cos\gamma+\cos\alpha\cos\beta\sin\gamma$$

24.
$$\cos(\alpha+\beta+\gamma)=\cos\alpha\cos\beta\cos\gamma-\cos\alpha\sin\beta\sin\gamma-\sin\alpha\cos\beta\sin\gamma-\sin\alpha\sin\beta\cos\gamma$$

25. $$\frac{\sin(\alpha\mp\beta)}{\sin\alpha\pm\sin\beta}=\frac{\sin\frac{1}{2}(\alpha\mp\beta)}{\sin\frac{1}{2}(\alpha\pm\beta)}$$

26. $$\frac{\sin(\alpha\mp\beta)}{\sin\alpha\mp\sin\beta}=\frac{\cos\frac{1}{2}(\alpha\mp\beta)}{\cos\frac{1}{2}(\alpha\pm\beta)}$$

NOTE. — Formulas (25) and (26) depend upon others subsequently given.

Functions of Periodic Values of the Arc or Angle.

[In the following formulas k is any integer positive or negative.]

27. $\sin k\pi = 0 \qquad \sin \frac{2k+1}{2}\pi = (-1)^k$

$\cos k\pi = (-1)^k \qquad \cos \frac{2k+1}{2}\pi = 0$

$\tan k\pi = 0 \qquad \tan \frac{2k+1}{2}\pi = \infty$

28.

$$\sin\varphi = \pm \sin(2k\pi \pm \varphi) = \mp \sin[(2k+1)\pi \pm \varphi]$$
$$= \mp \cos\left(\frac{4k+1}{2}\pi \pm \varphi\right) = \pm \cos\left(\frac{4k-1}{2}\pi \pm \varphi\right)$$

$\csc\varphi = \pm \csc(2k\pi \pm \varphi) = \text{\&c.}$

29.

$$\cos\varphi = \cos(2k\pi \pm \varphi) = -\cos[(2k+1)\pi \pm \varphi]$$
$$= \sin\left(\frac{4k+1}{2}\pi \pm \varphi\right) = -\sin\left(\frac{4k-1}{2}\pi \pm \varphi\right)$$

$\sec\varphi = \sec(2k\pi \pm \varphi) = \text{\&c.}$

30. $\tan\varphi = \pm \tan(k\pi \pm \varphi) = \mp \cot\left(\frac{2k+1}{2}\pi \pm \varphi\right)$

$\cot\varphi = \pm \cot(k\pi \pm \varphi) = \text{\&c.}$

Formulas (28), (29), and (30) give the only solutions of the equations

$\sin\varphi = \pm \sin\alpha,\quad \sin\varphi = \pm\cos\alpha,\quad \tan\varphi = \pm\tan\alpha,\quad \tan\varphi = \pm\cot\alpha,$

and of the equivalent equations

$\csc\varphi = \pm\csc\alpha,\quad \csc\varphi = \pm\sec\alpha,\quad \cot\varphi = \pm\cot\alpha,\quad \cot\varphi = \pm\tan\alpha.$

If *any two* of the six elementary functions (not reciprocals of each other) have equal values for φ and α, the only solution is

$$\varphi = 2k\cdot\pi + \alpha.$$

31.
$$\sin(\varphi \pm k\,360^\circ) = \sin\varphi$$
$$\cos(\varphi \pm k\,360^\circ) = \cos\varphi$$

32.
$$\sin(\varphi \pm 270^\circ) = \mp\cos\varphi$$
$$\cos(\varphi \pm 270^\circ) = \pm\sin\varphi$$

33.
$$\sin(\varphi \pm 180^\circ) = -\sin\varphi$$
$$\cos(\varphi \pm 180^\circ) = -\cos\varphi$$

34.
$$\sin(\varphi \pm 90^\circ) = \pm\cos\varphi$$
$$\cos(\varphi \pm 90^\circ) = \mp\sin\varphi$$

35.
$$\sin(90^\circ - \varphi) = \cos\varphi$$
$$\cos(90^\circ - \varphi) = \sin\varphi$$

36.
$$\sin(180^\circ - \varphi) = \sin\varphi$$
$$\cos(180^\circ - \varphi) = -\cos\varphi$$

37.
$$\sin(270^\circ - \varphi) = -\cos\varphi$$
$$\cos(270^\circ - \varphi) = -\sin\varphi$$

38.
$$\sin(360^\circ - \varphi) = -\sin\varphi$$
$$\cos(360^\circ - \varphi) = \cos\varphi$$

39.
$$\sin(-\varphi) = -\sin\varphi \qquad \cot(-\varphi) = -\cot\varphi$$
$$\cos(-\varphi) = \cos\varphi \qquad \sec(-\varphi) = \sec\varphi$$
$$\tan(-\varphi) = -\tan\varphi \qquad \csc(-\varphi) = -\csc\varphi$$

Sums and Products of Functions.

40. $\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$

41. $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$

42. $\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta)$

43. $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$

44. $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$
$= \cos^2\beta - \cos^2\alpha$

45. $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$
$= \cos^2\beta - \sin^2\alpha$

46. $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

47. $\sin\alpha - \sin\beta = 2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$

48. $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

49. $\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$

50. $$\frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}$$

51. $$\frac{\cos\alpha - \cos\beta}{\cos\alpha + \cos\beta} = \tan\tfrac{1}{2}(\beta+\alpha)\tan\tfrac{1}{2}(\beta-\alpha)$$

52. $$\frac{\sin\alpha \pm \sin\beta}{\cos\alpha + \cos\beta} = \tan\tfrac{1}{2}(\alpha\pm\beta)$$

53. $$\frac{\sin\alpha \mp \sin\beta}{\cos\beta - \cos\alpha} = \cot\tfrac{1}{2}(\alpha\pm\beta)$$

54. $$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha \pm \beta)}{\cos\alpha\cos\beta}$$

55. $$\cot\beta \pm \cot\alpha = \frac{\sin(\alpha \pm \beta)}{\sin\alpha\sin\beta}$$

56. $$\tan\alpha \pm \cot\beta = \frac{\pm\cos(\alpha \mp \beta)}{\cos\alpha\sin\beta}$$

57. $$\frac{\sin(\alpha \pm \beta)}{\sin(\alpha \mp \beta)} = \frac{\tan\alpha \pm \tan\beta}{\tan\alpha \mp \tan\beta} = \frac{\cot\beta \pm \cot\alpha}{\cot\beta \mp \cot\alpha}$$

58. $$\frac{\cos(\alpha \pm \beta)}{\cos(\alpha \mp \beta)} = \frac{1 \mp \tan\alpha\tan\beta}{1 \pm \tan\alpha\tan\beta} = \frac{\cot\beta \mp \tan\alpha}{\cot\beta \pm \tan\alpha}$$

59. $$\frac{\cos(\alpha \mp \beta)}{\sin(\alpha \pm \beta)} = \frac{\cot\beta \pm \tan\alpha}{\cot\beta\tan\alpha \pm 1}$$

Functions of Multiple Angles.

60. $\sin k\,\alpha = 2\sin(k-1)\,\alpha\cos\alpha - \sin(k-2)\,\alpha$

61. $\sin k\,\alpha = 2\cos(k-1)\,\alpha\sin\alpha + \sin(k-2)\,\alpha$

62. $\cos k\,\alpha = 2\cos(k-1)\,\alpha\cos\alpha - \cos(k-2)\,\alpha$

63. $\cos k\,\alpha = -2\sin(k-1)\,\alpha\sin\alpha + \cos(k-2)\,\alpha$

64. $$\tan k\,\alpha = \frac{\tan(k-1)\,\alpha + \tan\alpha}{1 - \tan(k-1)\,\alpha\tan\alpha}$$

65. $\sin 2\,\alpha = 2\sin\alpha\cos\alpha$

$\sin 3\,\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\sin 4\,\alpha = (4\sin\alpha - 8\sin^3\alpha)\cos\alpha$

$\sin 5\,\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha$

. . . . .

66. $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$
$= \cos^2\alpha - \sin^2\alpha$

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$

$\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$

. . . .

67. $$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2\cot\alpha}{\cot^2\alpha - 1} = \frac{2}{\cot\alpha - \tan\alpha}$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$$

. . . .

68. $$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha} = \frac{1 - \tan^2\alpha}{2\tan\alpha} = \frac{\cot\alpha - \tan\alpha}{2}$$

69. $$\sec 2\alpha = \frac{\sec^2\alpha}{1 - \tan^2\alpha} = \frac{\cot\alpha + \tan\alpha}{\cot\alpha - \tan\alpha}$$

70. $\csc 2\alpha = \frac{1}{2}\sec\alpha\csc\alpha = \frac{1}{2}(\tan\alpha + \cot\alpha)$

Functions of Half an Angle.

71. $$1 = \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}$$

72. $$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

73. $$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$$

74. $$1 + \sin\alpha = \left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)^2$$

75. $$1 - \sin\alpha = \left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)^2$$

76. $$1 + \cos\alpha = 2\cos^2\frac{\alpha}{2}$$

77. $$1 - \cos\alpha = 2\sin^2\frac{\alpha}{2}$$

78. $$\sin\frac{\alpha}{2} = \sqrt{\tfrac{1}{2}(1 - \cos\alpha)}$$
$$= \tfrac{1}{2}\sqrt{1 + \sin\alpha} - \tfrac{1}{2}\sqrt{1 - \sin\alpha}$$

79. $$\cos\frac{\alpha}{2} = \sqrt{\tfrac{1}{2}(1 + \cos\alpha)}$$
$$= \tfrac{1}{2}\sqrt{1 + \sin\alpha} + \tfrac{1}{2}\sqrt{1 - \sin\alpha}$$

80. $$\tan\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} = \frac{1 - \cos\alpha}{\sin\alpha}$$
$$= \frac{\sin\alpha}{1 + \cos\alpha} = \csc\alpha - \cot\alpha$$

81. $$\cot\frac{\alpha}{2} = \sqrt{\frac{1 + \cos\alpha}{1 - \cos\alpha}} = \frac{1 + \cos\alpha}{\sin\alpha}$$
$$= \frac{\sin\alpha}{1 - \cos\alpha} = \csc\alpha + \cot a$$

82. $$\sec\frac{\alpha}{2} = \sqrt{\frac{2\sec\alpha}{\sec\alpha + 1}}$$

83. $$\csc\frac{\alpha}{2} = \sqrt{\frac{2\sec\alpha}{\sec\alpha - 1}}$$

84. $$\tan\frac{\alpha}{2} = \frac{1 + \sin\alpha - \cos\alpha}{1 + \sin\alpha + \cos\alpha}$$

85. $$\sin\alpha + \cos\alpha = \sqrt{1 + \sin 2\alpha}$$

86. $$\sin\alpha - \cos\alpha = \sqrt{1 - \sin 2\alpha}$$

Formulas in which 45° or 30° appears

87. $1 + \sin\alpha = 2\sin^2\left(45° + \frac{\alpha}{2}\right) = 2\cos^2\left(45° - \frac{\alpha}{2}\right)$

88. $1 - \sin\alpha = 2\sin^2\left(45° - \frac{\alpha}{2}\right) = 2\cos^2\left(45° + \frac{\alpha}{2}\right)$

89. $\tan(45° \pm \alpha) = \cot(45° \mp \alpha) = \dfrac{1 \pm \tan\alpha}{1 \mp \tan\alpha}$

$$= \frac{\cos\alpha \pm \sin\alpha}{\cos\alpha \mp \sin\alpha} = \sqrt{\frac{1 \pm \sin 2\alpha}{1 \mp \sin 2\alpha}}$$

90. $\tan\left(45° \pm \frac{\alpha}{2}\right) = \cot\left(45° \mp \frac{\alpha}{2}\right) = \sqrt{\dfrac{1 \pm \sin\alpha}{1 \mp \sin\alpha}}$

$$= \frac{1 \pm \sin\alpha}{\cos\alpha} = \frac{\cos\alpha}{1 \mp \sin\alpha}$$

91. $\tan(\alpha - 45°) = \dfrac{\tan\alpha - 1}{\tan\alpha + 1}$

92. $\sin(45° + \alpha) = \dfrac{\sin\alpha + \cos\alpha}{\sqrt{2}} = \cos(45° - \alpha)$

93. $\cos(45° + \alpha) = \dfrac{\cos\alpha - \sin\alpha}{\sqrt{2}} = \sin(45° - \alpha)$

94. $\tan(45° + \alpha) + \tan(45° - \alpha) = 2\sec 2\alpha$

95. $\tan(45° + \alpha) - \tan(45° - \alpha) = 2\tan 2\alpha$

96. $\tan(45° + \alpha)\tan(45° - \alpha) = 1$

97. $\sin(30° + \alpha) + \sin(30° - \alpha) = \cos\alpha$

98. $\sin(30° + \alpha) - \sin(30° - \alpha) = \sin\alpha\sqrt{3}$

99. $\cos(30° + \alpha) + \cos(30° - \alpha) = \cos\alpha\sqrt{3}$

100. $\cos(30° + \alpha) - \cos(30° - \alpha) = -\sin\alpha$

Expressions equivalent to sin α.

101. $\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{(1+\cos\alpha)(1-\cos\alpha)}$

$$= \cos\alpha\tan\alpha = \frac{\cos\alpha}{\cot\alpha} = \frac{\tan\alpha}{\sec\alpha} = \frac{1}{\csc\alpha}$$

$$= \frac{\tan\alpha}{\sqrt{(1+\tan^2\alpha)}} = \frac{1}{\sqrt{(1+\cot^2\alpha)}} = \frac{\sqrt{(\sec^2\alpha-1)}}{\sec\alpha}$$

$$= \pm\sqrt{\tfrac{1}{2}(1-\cos 2\alpha)} = 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha$$

$$= \frac{2\tan\frac{1}{2}\alpha}{1+\tan^2\frac{1}{2}\alpha} = \frac{1}{\cot\frac{1}{2}\alpha-\cot\alpha} = \frac{1}{\tan\frac{1}{2}\alpha+\cot\alpha}$$

$$= \frac{2}{\tan\frac{1}{2}\alpha+\cot\frac{1}{2}\alpha} = 2\sin^2(45^\circ+\tfrac{1}{2}\alpha)-1$$

$$= 1-2\sin^2(45^\circ-\tfrac{1}{2}\alpha) = \frac{1-\tan^2(45^\circ-\frac{1}{2}\alpha)}{1+\tan^2(45^\circ-\frac{1}{2}\alpha)}$$

Expressions equivalent to cos α.

102. $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{(1+\sin\alpha)(1-\sin\alpha)}$

$$= \sin\alpha\cot\alpha = \frac{\sin\alpha}{\tan\alpha} = \frac{\cot\alpha}{\csc\alpha} = \frac{1}{\sec\alpha}$$

$$= \frac{\cot\alpha}{\sqrt{(1+\cot^2\alpha)}} = \frac{1}{\sqrt{(1+\tan^2\alpha)}} = \frac{\sqrt{(\csc^2\alpha-1)}}{\csc\alpha}$$

$$= \pm\sqrt{\tfrac{1}{2}(1+\cos 2\alpha)}$$

$$= \frac{\sin 2\alpha}{2\sin\alpha} = \cos^2\tfrac{1}{2}\alpha-\sin^2\tfrac{1}{2}\alpha = 1-2\sin^2\tfrac{1}{2}\alpha$$

$$= 2\cos^2\tfrac{1}{2}\alpha-1 = \frac{1-\tan^2\frac{1}{2}\alpha}{1+\tan^2\frac{1}{2}\alpha} = \frac{\cot^2\frac{1}{2}\alpha-1}{\cot^2\frac{1}{2}\alpha+1}$$

$$= \frac{\cot\frac{1}{2}\alpha-\tan\frac{1}{2}\alpha}{\cot\frac{1}{2}\alpha+\tan\frac{1}{2}\alpha} = \frac{1}{\tan\alpha\cot\frac{1}{2}\alpha-1}$$

$$= \frac{1}{1+\tan\alpha\tan\frac{1}{2}\alpha} = \frac{2}{\tan(45^\circ+\frac{1}{2}\alpha)+\cot(45^\circ+\frac{1}{2}\alpha)}$$

$$= 2\cos(45^\circ+\tfrac{1}{2}\alpha)\cos(45^\circ-\tfrac{1}{2}\alpha)$$

Expressions equivalent to tan α.

103. $$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1}{\cot\alpha} = \sqrt{\sec^2\alpha - 1}$$

$$= \frac{1}{\sqrt{(\csc^2\alpha - 1)}} = \frac{\sin\alpha}{\sqrt{(1-\sin^2\alpha)}} = \frac{\sqrt{(1-\cos^2\alpha)}}{\cos\alpha}$$

$$= \pm\sqrt{\frac{1-\cos 2\alpha}{1+\cos 2\alpha}} = \frac{\sin 2\alpha}{1+\cos 2\alpha} = \frac{1-\cos 2\alpha}{\sin 2\alpha}$$

$$= \csc 2\alpha - \cot 2\alpha = \cot\alpha - 2\cot 2\alpha$$

$$= \frac{2\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha} = \frac{2\cot\frac{1}{2}\alpha}{\cot^2\frac{1}{2}\alpha - 1}$$

$$= \frac{2}{\cot\frac{1}{2}\alpha - \tan\frac{1}{2}\alpha} = \frac{\tan(45^\circ + \frac{1}{2}\alpha) - \tan(45^\circ - \frac{1}{2}\alpha)}{2}$$

$$= \frac{\tan(45^\circ+\alpha) - 1}{\tan(45^\circ+\alpha)+1} = \frac{1-\tan(45^\circ-\alpha)}{1+\tan(45^\circ-\alpha)}$$

Expressions equivalent to cot α, sec α, *and* csc α, *may be found by taking the reciprocals of those above given for* tan α, cos α, *and* sin α, *respectively.*

III. Plane Triangles.

(a) General Properties.

104. $$\begin{cases} a = b \cos C + c \cos B \\ b = c \cos A + a \cos C \\ c = a \cos B + b \cos A \end{cases}$$

All the relations between the six parts of a triangle can be deduced from the three equations (**104**) by algebraic transformations.

105. $$A + B + C = 180°$$

106. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

107. $$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

108. $$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}$$

109. $$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}$$

110. $$\begin{cases} a^2 = b^2 + c^2 - 2\, b\, c \cos A \\ b^2 = c^2 + a^2 - 2\, c\, a \cos B \\ c^2 = a^2 + b^2 - 2\, a\, b \cos C \end{cases}$$

111. If ω denote an angle of a parallelogram, m and n its sides, and d the diagonal which divides ω,

$$d^2 = m^2 + n^2 + 2\, m\, n \cos \omega.$$

112. $$s = \tfrac{1}{2}(a + b + c)$$

113. $$\left\{ {}^{*}\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}} \right.$$

114. $$\left\{ \cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}} \right.$$

115. $$\left\{ \tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \right.$$

116. Let r denote the radius of the inscribed circle.

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= s \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C$$

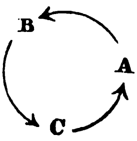
117. $$\left\{ \tan \tfrac{1}{2} A = \frac{r}{s-a} \right.$$

118. Let T denote the area of the triangle.

$$T = s\,r = \sqrt{s(s-a)(s-b)(s-c)}$$

119. $$\left\{ \sin A = \frac{2\,T}{b\,c} = \frac{2\,s\,r}{b\,c} \right.$$

* The character $\{$ indicates a set of three equations, of which only one is printed. The others are obtained by advancing the letters, that is, by substituting for each letter of the equation the next letter in the fixed order indicated by the figure in the margin. In this manner the second equation of (**104**) may be obtained from the first, and then the third from the second.



120. $$T = \frac{c^2 \sin A \sin B}{2 \sin (A + B)} = \tfrac{1}{2}\, a\, b \sin C$$

121. Let p_a, p_b, and p_c denote perpendiculars from the angles upon the sides a, b, and c respectively.

$$\left\{ p_a = b \sin C = c \sin B = \frac{b\,c}{a} \sin A \right.$$
$$= \frac{\sin B \sin C}{\sin A}\, a = \frac{2\,T}{a}$$

122. Let r_a, r_b, and r_c denote the radii of the escribed circles touching the sides a, b, and c respectively.

$$\left\{ r_a = s \tan \tfrac{1}{2} A = \frac{T}{s - a} = \frac{s\,r}{s - a} \right.$$

123. $$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} = \frac{1}{p_a} + \frac{1}{p_b} + \frac{1}{p_c}$$

124. Let R denote the radius of the circumscribed circle.

$$R = \tfrac{1}{2}\, a \csc A = \tfrac{1}{4}\, s \sec \tfrac{1}{2} A \sec \tfrac{1}{2} B \sec \tfrac{1}{2} C$$
$$= \tfrac{1}{4}(r_a + r_b + r_c - r) = \frac{a\,b\,c}{4\,T}$$

(b) Solutions.

Right Triangles.

125. Given an angle and the hypotenuse, A and h.

$$B = 90^\circ - A$$

$$a = h \sin A \qquad \log a = \log h + \log \sin A$$
$$b = h \cos A \qquad \log b = \log h + \log \cos A$$

126. Given an angle and the leg opposite, A and a.

$$B = 90^\circ - A$$

$$h = a \csc A \qquad \log h = \log a + \log \csc A$$
$$b = a \cot A \qquad \log b = \log a + \log \cot A$$

127. Given an angle and the leg adjacent, A and b.

$$B = 90^\circ - A$$

$$h = b \sec A \qquad \log h = \log b + \log \sec A$$
$$a = b \tan A \qquad \log a = \log b + \log \tan A$$

128. Given the hypotenuse and a leg, h and a.

$$\sin A = \cos B = \frac{a}{h} \qquad b = \sqrt{(h + a)\,(h - a)}$$

$$\log \sin A = \log \cos B = \log a + \text{co-log}\, h$$
$$\log b = \tfrac{1}{2}\,[\log\,(h + a) + \log\,(h - a)]$$

129. Given the two legs, a and b.

$$\tan A = \cot B = \frac{a}{b} \qquad h = a \csc A$$

$$\log \tan A = \log \cot B = \log a + \text{co-log}\, b$$
$$\log h = \log a + \log \csc A$$
$$\textit{or } \log h = \log b + \tfrac{1}{2}\,\mathfrak{G}\,(2 \log \tan A)$$

130. For accurate computation the following additional formulas *Right Triangles* are useful.

(1.) $$h - b = 2h \sin^2 \tfrac{1}{2} A$$

which gives $h - b$ with accuracy when A is small, or A with accuracy when h and b are nearly equal.

(2.) $$\tan \tfrac{1}{2} A = \sqrt{\frac{h-b}{h+b}} = \frac{h-b}{a}$$

(3.) $$\sin (45^\circ \pm \tfrac{1}{2} A) = \sqrt{\frac{h \pm a}{2h}}$$

(4.) $$\cos (45^\circ \pm \tfrac{1}{2} A) = \sqrt{\frac{h \mp a}{2h}}$$

(5.) $$\tan (45^\circ \pm \tfrac{1}{2} A) = \sqrt{\frac{h \pm a}{h \mp}}$$

(6.) $$\tan (45^\circ \pm A) = \frac{b \pm a}{b \mp a}$$

(7.) $$\sin (B - A) = \frac{(b+a)(b-a)}{h^2}$$

(8.) $$\cos (B - A) = \frac{2ab}{h^2}$$

(9.) $$\tan (B - A) = \frac{(b+a)(b-a)}{2ab}$$

which gives $B - A$ with great accuracy when a and b are nearly equal.

(10.) $$S = \log \left(\frac{\sin \varphi}{\varphi}\right) \quad T = \log \left(\frac{\tan \varphi}{\varphi}\right)$$

The quantities S and T for small arcs are usually given in the tables. Their use is an additional means of securing accuracy in the results.

Oblique Triangles.

131. Case I. Given two angles and a side, A, B, and a.

First Solution. $C = 180° - (A + B)$

$$\log b = \log a + \log \sin B + \log \csc A$$

$$\log c = \log a + \log \sin C + \log \csc A$$

Second Solution. $C = 180° - (A + B)$

$$\log (b + c) = \log a + \log \cos \tfrac{1}{2} (B - C) + \log \sec \tfrac{1}{2} (B + C)$$

$$\log (b - c) = \log a + \log \sin \tfrac{1}{2} (B - C) + \log \csc \tfrac{1}{2} (B + C)$$

$$b = \tfrac{1}{2} (b + c) + \tfrac{1}{2} (b - c)$$

$$c = \tfrac{1}{2} (b + c) - \tfrac{1}{2} (b - c)$$

Third Solution. — When A and B (and consequently a and b) are nearly equal, the difference between a and b is computed with great accuracy by the formula

$$a - b = \frac{2\, a \cos \tfrac{1}{2} (A + B) \sin \tfrac{1}{2} (A - B)}{\sin A}$$

which is obtained by combining (**106**) with (**47**).

132. Case II. Given two sides and an angle opposite one of them, a, b, and A.

First Solution.

(1.) $\log \sin B = \text{co-log}\, a + \log b + \log \sin A$

giving supplementary values of B.

Let B_1 be the acute value, and B_2 the obtuse value.

(2.) $C_1 = B_2 - A \qquad\qquad C_2 = B_1 - A$

$$(3.)\quad \begin{cases} \log c_1 = \log a + \log \sin C_1 + \log \csc A \\ \log c_2 = \log a + \log \sin C_2 + \log \csc A \end{cases}$$

If (1) gives $\sin B > 1$, B is imaginary, and there is no solution.

If (2) gives a negative value of C_1 or C_2, the corresponding solution may be rejected; or, in problems which admit the interpretation of negative lines and angles, may be completed with proper regard to the signs.

Second Solution. — Draw a perpendicular to the side c from the opposite angle, and denote the segments of c, adjacent to the angles A and B, by c_{A} and c_{B} respectively.

(1.) Find the angles B and C as in the *First Solution.*

(2.) $c_{\text{A}} = b \cos A, \quad c_{\text{B}} = a \cos B, \quad c = c_{\text{A}} + c_{\text{B}},$

which gives two values of c; for c_{B} is positive or negative according as B is acute or obtuse.

133. Case III. Given two sides and the included angle, a, b, and C.

First Solution.

(1.) $\frac{1}{2}(A + B) = 90° - \frac{1}{2}C$

(2.) $a + b : a - b = \tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B)$

$\log \tan \frac{1}{2}(A - B) = \text{co-log}\,(a + b) + \log (a - b) + \log \cot \frac{1}{2} C$

(3.) $\begin{cases} A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B) \\ B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B) \end{cases}$

(4.) $\begin{cases} \log c = \log a + \log \sin C + \log \csc A \\ \log c = \log b + \log \sin C + \log \csc B \end{cases}$

A more convenient mode of finding c is by one of the following equations: —

$$\log c = \log (a + b) + \log \cos \tfrac{1}{2}(A + B) + \log \sec \tfrac{1}{2}(A - B)$$

$$\log c = \log (a - b) + \log \sin \tfrac{1}{2}(A + B) + \log \csc \tfrac{1}{2}(A - B)$$

Second Solution. — Given C and the logarithms of a and b.

Let λ be an auxiliary angle such that $\tan \lambda = \frac{a}{b}$.

(1.) $$\log \tan \lambda = \log a - \log b$$

(2.)

$$\log \tan \tfrac{1}{2}(A - B) = \log \tan (\lambda - 45^\circ) + \log \tan \tfrac{1}{2}(A + B)$$

The rest of the solution by equations already given.

Third Solution. — To find A, B, and c directly from the data.

(1.) $$\tan A = \frac{a \sin C}{b - a \cos C}$$

(2.) $$\tan B = \frac{b \sin C}{a - b \cos C}$$

(3.) $$\begin{cases} \tan \lambda = \dfrac{2 \sin \frac{1}{2} C}{a - b} \sqrt{a\,b} \\ c = (a - b) \sec \lambda \end{cases}$$

134. CASE IV. Given the three sides, a, b, and c.

$$s = \tfrac{1}{2}(a + b + c)$$

$$\log r = \tfrac{1}{2}[\text{co-log}\, s + \log (s - a) + \log (s - b) + \log (s - c)]$$

$$\log \tan \tfrac{1}{2} A = \log r - \log (s - a)$$

$$\log \tan \tfrac{1}{2} B = \log r - \log (s - b)$$

$$\log \tan \tfrac{1}{2} C = \log r - \log (s - c)$$

Other methods of solution are furnished by formulas (**113**) and (**114**), of which (**113**) is to be preferred when the half-angle is less than 45°, and (**114**) when the half-angle is more than 45°. The solution here given, however, is as simple as any other, when only one angle is required, and much simpler when all the angles are required. It has the advantage of being accurate for all values of the angles.

IV. Spherical Triangles.

(a) General Properties.

135. $$\begin{cases}\cos a = \cos b \cos c + \sin b \sin c \cos A \\ \cos b = \cos c \cos a + \sin c \sin a \cos B \\ \cos c = \cos a \cos b + \sin a \sin b \cos C\end{cases}$$

All the relations between the six parts of a spherical triangle can be deduced from these three equations by algebraic transformations.

136. $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

To deduce these equations from **(135)**.

Eliminate c from the first two equations of the set. This is done by addition and subtraction, whereby

$$(\cos a + \cos b)(1 - \cos c) = \sin c (\sin b \cos A + \sin a \cos B)$$
$$(\cos a - \cos b)(1 + \cos c) = \sin c (\sin b \cos A - \sin a \cos B)$$

whence, by multiplication and reduction,

$$\sin^2 a \sin^2 B = \sin^2 b \sin^2 A$$

whence the first equation of **(136)**.

137. $$\begin{cases}\cos A = -\cos B \cos C + \sin B \sin C \cos a. \\ \cos B = -\cos C \cos A + \sin C \sin A \cos b. \\ \cos C = -\cos A \cos B + \sin A \sin B \cos c.\end{cases}$$

To deduce these equations from **(135)**.

Eliminate b and c from the first two equations of the set. This is done by substituting for $\cos c$ its value taken from the third equation; substituting for $\sin a$, $\sin b$, and $\sin c$, the proportional quantities $\sin A$, $\sin B$, and $\sin C$; and eliminating $\cos b$ from the results. This gives the first equation of the present set.

Equations **(135)** applied to the polar triangle give **(137)** at once.

$$
\textbf{138.}\quad \begin{cases}
\cot a \sin b = \cos b \cos C + \sin C \cot A \\
\cot b \sin c = \cos c \cos A + \sin A \cot B \\
\cot c \sin a = \cos a \cos B + \sin B \cot C \\
\\
\cot a \sin c = \cos c \cos B + \sin B \cot A \\
\cot b \sin a = \cos a \cos C + \sin C \cot B \\
\cot c \sin b = \cos b \cos A + \sin A \cot C
\end{cases}
$$

These equations are best established by deducing them from (**135**). In the first equation of that set substitute for $\cos c$ its value taken from the third equation, and for $\sin c$ its value, $\sin a \dfrac{\sin C}{\sin A}$, taken from (**136**). The result, reduced, is the first equation of the present set.

$$
\textbf{139.}\quad \begin{cases}
\sin a \cos B = \sin c \cos b - \cos c \sin b \cos A \\
\sin b \cos C = \sin a \cos c - \cos a \sin c \cos B \\
\sin c \cos A = \sin b \cos a - \cos b \sin a \cos C \\
\\
\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A \\
\sin b \cos A = \sin c \cos a - \cos c \sin a \cos B \\
\sin c \cos B = \sin a \cos b - \cos a \sin b \cos C
\end{cases}
$$

To deduce these equations from (**135**). Multiply the first equation of that set by $\cos c$; substitute for $\cos c \cos a$ its value taken from the second equation, and reduce.

$$
\textbf{140.}\quad \begin{cases}
\sin A \cos b = \sin C \cos B + \cos C \sin B \cos a \\
\sin B \cos c = \sin A \cos C + \cos A \sin C \cos b \\
\sin C \cos a = \sin B \cos A + \cos B \sin A \cos c \\
\\
\sin A \cos c = \sin B \cos C + \cos B \sin C \cos a \\
\sin B \cos a = \sin C \cos A + \cos C \sin A \cos b \\
\sin C \cos b = \sin A \cos B + \cos A \sin B \cos c
\end{cases}
$$

Obtained by applying (**139**) to the Polar Triangle.

141. $$\begin{cases} \cos a = \cos(b+c) + 2 \sin b \sin c \cos^2 \tfrac{1}{2} A \\ \quad = \cos(b-c) - 2 \sin b \sin c \sin^2 \tfrac{1}{2} A \end{cases}$$

142. $$\begin{cases} \cos A = -\cos(B+C) - 2 \sin B \sin C \sin^2 \tfrac{1}{2} a \\ \quad = -\cos(B-C) + 2 \sin B \sin C \cos^2 \tfrac{1}{2} a \end{cases}$$

143. $$s = \tfrac{1}{2}(a+b+c)$$

144. $$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$$

145. $$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$$

146. $$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$$

147. $$S = \tfrac{1}{2}(A+B+C)$$

148. $$2E = A + B + C - 180^\circ = 2S - 180^\circ$$
$$= \text{spherical excess of the triangle.}$$

149. $$\cos \tfrac{1}{2} a = \sqrt{\frac{\sin(B-E)\sin(C-E)}{\sin B \sin C}}$$

150. $$\sin \tfrac{1}{2} a = \sqrt{\frac{\sin E \sin(A-E)}{\sin B \sin C}}$$

151. $$\cot \tfrac{1}{2} a = \sqrt{\frac{\sin(B-E)\sin(C-E)}{\sin E \sin(A-E)}}$$

Gauss's Equations.

152. $$\left\{ \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2} C} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2} c} \right. \quad . \ . \quad \text{I.}$$

153. $$\left\{ \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2} c} \right. \quad . \ . \quad \text{II.}$$

154. $$\left\{ \frac{\cos \frac{1}{2}(A+B)}{\sin \frac{1}{2} C} = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2} c} \right. \quad . \ . \quad \text{III.}$$

155. $$\left\{ \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C} = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2} c} \right. \quad . \ . \quad \text{IV.}$$

Napier's Analogies.

156. $$\left\{ \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a-b)} \right. \quad . \ . \quad \text{I.}$$

157. $$\left\{ \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a+b)} \right. \quad . \ . \quad \text{II.}$$

158. $$\left\{ \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A-B)} \right. \quad . \ . \quad \text{III.}$$

159. $$\left\{ \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A+B)} \right. \quad . \ . \quad \text{IV.}$$

The equation

$$\sin \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B + \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B,$$

reduced by (**144**) and (**145**), gives

$$\sin \tfrac{1}{2}(A+B) = \frac{\sin(s-a) + \sin(s-b)}{\sin c} \cos \tfrac{1}{2} C;$$

which, reduced by (**143**) and (**46**), gives (**152**).

The other three of Gauss's Equations may be proved in a similar manner. Napier's Analogies follow directly from Gauss's Equations.

160. $\tan\frac{E}{2}=\sqrt{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}}$

161. Let r denote the polar radius of the circle inscribed in the triangle.

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

162. Let R denote the polar radius of the circle circumscribed about the triangle.

$$\cot R=\sqrt{\frac{\sin(A-E)\sin(B-E)\sin(C-E)}{\sin E}}$$

163. $\left\{\tan\frac{1}{2}A=\frac{\tan r}{\sin(s-a)}\right.$

164. $\left\{\cot\frac{1}{2}a=\frac{\cot R}{\sin(A-E)}\right.$

165.
$n=\sin s\tan r=\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}$

166.
$N=\sin E\cot R=\sqrt{\sin E\sin(A-E)\sin(B-E)\sin(C-E)}$

167. $\left\{\sin A=\frac{2n}{\sin b\sin c}=\frac{2\sin s\tan r}{\sin b\sin c}\right.$

168. $\left\{\sin a=\frac{2N}{\sin B\sin C}=\frac{2\sin E\cot R}{\sin B\sin C}\right.$

169. $\tan E=\frac{\tan\frac{1}{2}a\tan\frac{1}{2}b\sin C}{1+\tan\frac{1}{2}a\tan\frac{1}{2}b\cos C}$

$$=\frac{2\sin s\tan r}{1+\cos a+\cos b+\cos c}$$

170. Let p_a, p_b, and p_c denote perpendicular arcs upon the sides a, b, c, respectively.

$$\left\{\begin{aligned}\sin p_a &= \sin b \sin C = \sin c \sin B = \frac{\sin b \sin c}{\sin a} \sin A \\ &= \frac{\sin B \sin C}{\sin A} \sin a = \frac{2\,n}{\sin a} = \frac{2\,N}{\sin A}\end{aligned}\right.$$

171. Let r_a, r_b, and r_c denote the radii of the circles inscribed in the associated triangles (formed by prolonging the sides of the original triangle), and R_a, R_b, and R_c the radii of the circles circumscribed about the same triangles.

$$\left\{\begin{aligned}\tan r_a &= \sin s \tan \tfrac{1}{2} A = \frac{n}{\sin (s - a)} \\ &= \frac{\sin s \tan r}{\sin (s - a)}\end{aligned}\right.$$

172. $$\left\{\begin{aligned}\cot R_a &= \sin E \cot \tfrac{1}{2} a = \frac{N}{\sin (A - E)} \\ &= \frac{\sin E \cot R}{\sin (A - E)}\end{aligned}\right.$$

173.

$$\sin a \sin b + \cos a \cos b \cos C = \sin A \sin B - \cos A \cos B \cos c$$

174. The perpendicular arc CP drawn in a spherical triangle, coincides with the perpendicular arc C′P′ drawn in the polar triangle. Adopting a notation analogous to that of (**132**), we have

$$C_a + c_{A'} = c_A + C'_{a'} = C_b + c'_{B'} = c_B + C'_{b'} = 90^\circ$$

$$c = c_A + c_B \qquad\qquad C = C_a + C_b$$

(b) Properties of Right Triangles.

The following ten equations may be established geometrically, or deduced directly from **(135)** – **(140)** by making C a right angle and writing h for c.

175. $$\sin a = \sin h \sin A$$

176. $$\sin b = \sin h \sin B$$

177. $$\sin a = \tan b \cot B$$

178. $$\sin b = \tan a \cot A$$

179. $$\cos A = \cos a \sin B$$

180. $$\cos B = \cos b \sin A$$

181. $$\cos A = \cot h \tan b$$

182. $$\cos B = \cot h \tan a.$$

183. $$\cos h = \cos a \cos b$$

184. $$\cos h = \cot A \cot B$$

Napier's Rules.

Excluding the right angle from consideration, there are, in a right triangle, five parts, which may be regarded as forming a ring, in which each part has two parts *adjacent* to it, and two parts not adjacent, or *opposite* to it. If, instead of the hypotenuse and the two adjacent angles, their complements be used as parts, the ten equations, **(175)** – **(184)**, may be embodied in the following mnemonic rules: —

I. *The sine of any part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of any part is equal to the product of the cosines of the opposite parts.*

(c) Solutions.

Spherical Right Triangles.

185. Given an angle and the hypotenuse, A and h.

$$\sin a = \sin h \sin A$$
$$\tan b = \tan h \cos A$$
$$\cot B = \cos h \tan A$$
Check. $$\sin a = \tan b \cot B$$

$$a_2 = 180° - a_1$$
$$b_2 = 180° + b_1$$
$$B_2 = 180° + B_1$$

186. Given an angle and the leg opposite, A and a.

$$\sin h = \sin a \csc A$$
$$\sin b = \tan a \cot A$$
$$\sin B = \sec a \cos A$$
Check. $$\sin b = \sin h \sin B$$

$$h_2 = 180° - h_1$$
$$b_2 = 180° - b_1$$
$$B_2 = 180° - B_1$$

187. Given an angle and the leg adjacent, A and b

$$\tan a = \sin b \tan A$$
$$\cot h = \cot b \cos A$$
$$\cos B = \cos b \sin A$$
Check. $$\cos B = \tan a \cot h$$

$$a_2 = 180° + a_1$$
$$h_2 = 180° + h_1$$
$$B_2 = 360° - B_1$$

188. Given the hypotenuse and a leg, h and a.

$$\sin A = \sin a \csc h$$
$$\cos b = \sec a \cos h$$
$$\cos B = \tan a \cot h$$

Check. $\cos B = \sin A \cos b$

$$A_2 = 180° - A_1$$
$$b_2 = 360° - b_1$$
$$B_2 = 360° - B_1$$

189. Given the two legs, a and b.

$$\cos h = \cos a \cos b$$
$$\cot A = \cot a \sin b$$
$$\cot B = \sin a \cot b$$

Check. $\cos h = \cot A \cot B$

$$h_2 = 360° - h_1$$
$$A_2 = 180° + A_1$$
$$B_2 = 180° + B_1$$

190. Given the two angles, A and B.

$$\cos h = \cot A \cot B$$
$$\cos a = \cos A \csc B$$
$$\cos b = \csc A \cos B$$

Check. $\cos h = \cos a \cos b$

$$h_2 = 360° - h_1$$
$$a_2 = 360° - a_1$$
$$b_2 = 360° - b_1$$

191. Formulas for accurate computation in particular cases. See **130,** (10.)

(1.) When A is near 90°,

$$\tan\left(45° - \tfrac{1}{2}A\right) = \sqrt{\frac{\tan\frac{1}{2}(h-a)}{\tan\frac{1}{2}(h+a)}} = \sqrt{\tan\tfrac{1}{2}(B+b)\tan\tfrac{1}{2}(B-b)}$$

(2.) When A is small or near 180°,

$$\tan\tfrac{1}{2}A = \sqrt{\frac{\sin(h-b)}{\sin(h+b)}}$$

(3.) When a is near 90°,

$$\tan\left(45° - \tfrac{1}{2}a\right) = \sqrt{\frac{\sin(B-b)}{\sin(B+b)}}$$

(4.) When a is small or near 180°,

$$\tan\tfrac{1}{2}a = \sqrt{\tan\tfrac{1}{2}(h+b)\tan\tfrac{1}{2}(h-b)}$$

(5.) When h is near 90°,

$$\tan\left(45° - \tfrac{1}{2}h\right) = \sqrt{\frac{\tan\frac{1}{2}(A-a)}{\tan\frac{1}{2}(A+a)}}$$

(6.) When h is small,

$$\sin\tfrac{1}{2}h = \sqrt{\frac{-\cos(A+B)}{2\sin A\sin B}}$$

(7.) When h is near 180°,

$$\cos\tfrac{1}{2}h = \sqrt{\frac{\cos(A-B)}{2\sin A\sin B}}$$

Oblique Spherical Triangles.

Only those Spherical Triangles whose sides and angles are less than 360° are considered in the following solutions.

192. CASE I. Given two sides and the included angle, a, b, and C.

Two triangles always possible.

$$c_2 = 360° - c_1$$
$$A_2 = 180° + A_1$$
$$B_2 = 180° + B_1$$

First Solution.

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

$$\cot A = \frac{\cos a \sin b - \sin a \cos b . \cos C}{\sin a \sin C}$$

$$\cot B = \frac{\sin a \cos b - \cos a \sin b \cos C}{\sin b \sin C}$$

Second Solution, by Gauss's Equations.

Third Solution.

$$\tan b_C = \tan a \cos C$$

$$b_A = b - b_C$$

$$\cos c = \cos a \sec b_C \cos b_A$$

$$\cot A = \cot C \csc b_C \sin b_A$$

$$\sin B = \sin b \sin C \csc c$$
$$= \sin b \sin A \csc a$$

193. CASE II. Given one side and the two adjacent angles, A, B, and c.

Two triangles always possible.

$$C_2 = 360° - C_1$$
$$a_2 = 180° + a_1$$
$$b_2 = 180° + b_1$$

First Solution.

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

$$\cot a = \frac{\cos A \sin B + \sin A \cos B \cos c}{\sin A \sin c}$$

$$\cot b = \frac{\sin A \cos B + \cos A \sin B \cos c}{\sin B \sin c}$$

Second Solution, by Gauss's Equations.

Third Solution.

$$\cot B_c = \tan A \cos c$$

$$B_a = B - B_c$$

$$\cos C = \cos A \csc B_c \sin B_a$$

$$\cot a = \cot c \sec B_c \cos B_a$$

$$\sin b = \sin B \sin c \csc C$$
$$= \sin B \sin a \csc A$$

194. Case III. Given two sides and an angle opposite, one of them, a, b, and A.

Two triangles,

both possible when $\sin a > \sin b \sin A$,
both impossible when $\sin a < \sin b \sin A$,
identical when $\sin a = \sin b \sin A$.

First Solution.

$$(B) \quad \begin{cases} \sin B = \csc a \sin b \sin A \\ B_2 = 180^\circ - B_1 \end{cases}$$

$$(c) \quad \begin{cases} \tan c_A = \tan b \cos A \\ \cos (c - c_A) = \cos a \sec b \cos c_A \\ c_1 = c_A + (c - c_A)_1 \\ c_2 = c_A + (c - c_A)_2 \end{cases}$$

$$(C) \quad \begin{cases} \tan C_b = \sec b \cot A \\ \cos (C - C_b) = \cot a \tan b \cos C_b \\ C_1 = C_b + (C - C_b)_1 \\ C_2 = C_b + (C - C_b)_2 \end{cases}$$

Second Solution. — Find B as in the First Solution, and substitute its values separately in Gauss's Equations. Two values of c, and two of C will, in general, result for each value of B; of which those that exceed 360° may be set aside.

195. CASE IV. Given two angles and a side opposite one of them, A, B, and a.

Two triangles,

both possible when $\sin A > \sin B \sin a$,
both impossible when $\sin A < \sin B \sin a$,
identical when $\sin A = \sin B \sin a$.

First Solution.

$$(b) \quad \begin{cases} \sin b = \csc A \sin B \sin a \\ b_2 = 180^\circ - b_1 \end{cases}$$

$$(C) \quad \begin{cases} \cot C_a = \tan B \cos a \\ \sin (C - C_a) = \cos A \sec B \sin C_a \\ C_1 = C_a + (C - C_a)_1 \\ C_2 = C_a + (C - C_a)_2 \end{cases}$$

$$(c) \quad \begin{cases} \cot c_B = \sec B \cot a \\ \sin (c - c_B) = \cot A \tan B \sin c_B \\ c_1 = c_B + (c - c_B)_1 \\ c_2 = c_B + (c - c_B)_2 \end{cases}$$

Second Solution. — Find b as in the First Solution, and substitute its values separately in Gauss's Equations. Two values of C, and two of c will, in general, result for each value of b; of which those that exceed 360° may be set aside.

196. Case V. Given the three sides.

Two triangles such that

$$A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = 360°.$$

First Solution.

$$\left\{ \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \right.$$

Second Solution.

Compute halves of the angles

by the sines, **(144)**,
or by the cosines, **(145)**,
or by the tangents, **(146)**.

Instead of (**146**) it will often be more convenient to use (**161**) and (**163**).

197. Case VI. Given the three angles.

Two triangles such that

$$a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = 360°.$$

First Solution.

$$\left\{ \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \right.$$

Second Solution.

Compute halves of the sides

by the sines, **(150)**,
or by the cosines, **(149)**,
or by the cotangents, **(151)**.

Instead of (**151**) it will often be more convenient to use (**162**) and (**164**).

V. Inverse Circular Functions.

198. $$\text{arc sin } x = \text{arc cos } \sqrt{1-x^2} = \text{arc tan } \frac{x}{\sqrt{1-x^2}}$$
$$= \text{arc cot } \sqrt{\frac{1}{x^2}-1} = \text{arc sec } \frac{1}{\sqrt{1-x^2}} = \text{arc csc } \frac{1}{x}$$
$$= 2 \text{ arc sin } \sqrt{\tfrac{1}{2}\left(1-\sqrt{1-x^2}\right)} = \tfrac{1}{2} \text{ arc sin } 2x\sqrt{1-x^2}$$
$$= 2 \text{ arc tan } \frac{1-\sqrt{1-x^2}}{x} = \tfrac{1}{2} \text{ arc tan } \frac{2x\sqrt{1-x^2}}{1-2x^2}$$

199. $$\text{arc cos } x = \text{arc sin } \sqrt{1-x^2} = \text{arc tan } \sqrt{\frac{1}{x^2}-1}$$
$$= \text{arc cot } \frac{x}{\sqrt{1-x^2}} = \text{arc sec } \frac{1}{x} = \text{arc csc } \frac{1}{\sqrt{1-x^2}}$$
$$= 2 \text{ arc cos } \sqrt{\tfrac{1}{2}(1+x)} = \tfrac{1}{2} \text{ arc cos } (2x^2-1)$$
$$= 2 \text{ arc tan } \sqrt{\frac{1-x}{1+x}}$$

200. $$\text{arc tan } x = \text{arc sin } \frac{x}{\sqrt{1+x^2}} = \text{arc cos } \frac{1}{\sqrt{1+x^2}} = \text{arc cot } \frac{1}{x}$$
$$= \text{arc sec} \sqrt{1+x^2} = \text{arc csc } \sqrt{\frac{1}{x^2}+1}$$
$$= 2 \text{ arc tan } \frac{-1+\sqrt{1+x^2}}{x} = \tfrac{1}{2} \text{ arc tan } \frac{2x}{1-x^2}$$
$$= \tfrac{1}{2} \text{ arc sin } \frac{2x}{1+x^2} = \tfrac{1}{2} \text{ arc cos } \frac{1-x^2}{1+x^2}$$

201. $$\text{arc sin } x \pm \text{arc sin } y = \text{arc sin } \left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$$

202. $$\text{arc cos } x \pm \text{arc cos } y = \text{arc cos } \left(xy \mp \sqrt{(1-x^2)(1-y^2)}\right)$$

203. $$\text{arc tan } x \pm \text{arc tan } y = \text{arc tan } \frac{x \pm y}{1 \mp xy}$$

VI. Exponential Functions.

In the following formulas e is used to denote the base of the Naperian system of logarithms.

Also i is used as an abbreviation of $+\sqrt{-1}$, so that if n be any integer positive or negative or zero, we have

204. $\begin{cases} i^{4n} = +1 & i^{4n+2} = -1 \\ i^{4n+1} = +\sqrt{-1} & i^{4n+3} = -\sqrt{-1} \end{cases}$

205. $$\sin\varphi = \frac{e^{\varphi i} - e^{-\varphi i}}{2i}$$

206. $$\cos\varphi = \frac{e^{\varphi i} + e^{-\varphi i}}{2}$$

207. $$e^{\varphi i} = \cos\varphi + i\sin\varphi$$

208. $$e^{-\varphi i} = \cos\varphi - i\sin\varphi$$

209. $$e^{k\varphi i} = \cos k\varphi + i\sin k\varphi$$

210. $$e^{2k\pi i} = 1 \qquad e^{(2k+1)\pi i} = -1$$
$$e^{\frac{1}{2}(4k+1)\pi i} = i \qquad e^{\frac{1}{2}(4k+3)\pi i} = -i$$

211. $$e^{\varphi + 2k\pi i} = e^{\varphi}$$
$$e^{\varphi + (2k+1)\pi i} = -e^{\varphi}$$

212. $$(\cos\varphi \pm \mathrm{i}\sin\varphi)^n = \cos n\varphi \pm \mathrm{i}\sin n\varphi$$

213. $$\sqrt[n]{\cos\varphi \pm \mathrm{i}\sin\varphi} = \cos\frac{\varphi + 2k\pi}{n} \pm \mathrm{i}\sin\frac{\varphi + 2k\pi}{n}$$

214. Let z be any variable real or imaginary, and let r be its modulus, φ its argument.

$$z = r(\cos\varphi + \mathrm{i}\sin\varphi) = r\mathfrak{S}^{\varphi\mathrm{i}} = r\mathfrak{S}^{\varphi\mathrm{i} + 2k\pi\mathrm{i}}$$

$$\log z = \log r + (\varphi + 2k\pi)\,\mathrm{i}$$

215. $$\operatorname{arc}\sin x = -\mathrm{i}\log\left(\mathrm{i}x \pm \sqrt{1 - x^2}\right)$$

216. $$\operatorname{arc}\cos x = -\mathrm{i}\log\left(x \pm \mathrm{i}\sqrt{1 - x^2}\right)$$

217. $$\operatorname{arc}\tan x = \frac{1}{2\mathrm{i}}\log\frac{1 + \mathrm{i}x}{1 - \mathrm{i}x}$$

218. $$\sin\varphi\mathrm{i} = \tfrac{1}{2}\mathrm{i}\left(\mathfrak{S}^{\varphi} - \mathfrak{S}^{-\varphi}\right)$$

219. $$\cos\varphi\mathrm{i} = \tfrac{1}{2}\left(\mathfrak{S}^{\varphi} + \mathfrak{S}^{-\varphi}\right)$$

220. $$\sin(\alpha + \beta\mathrm{i}) = \tfrac{1}{2}(\mathfrak{S}^{\beta} + \mathfrak{S}^{-\beta})\sin\alpha + \tfrac{1}{2}(\mathfrak{S}^{\beta} - \mathfrak{S}^{-\beta})\,\mathrm{i}\cos\alpha$$

221. $$\cos(\alpha + \beta\mathrm{i}) = \tfrac{1}{2}(\mathfrak{S}^{\beta} + \mathfrak{S}^{-\beta})\cos\alpha - \tfrac{1}{2}(\mathfrak{S}^{\beta} - \mathfrak{S}^{-\beta})\,\mathrm{i}\sin\alpha$$

VII. Derivatives.

222. $D_\varphi \sin \varphi = \cos \varphi$

223. $D_\varphi \cos \varphi = - \sin \varphi$

224. $D_\varphi \tan \varphi = 1 + \tan^2 \varphi$

225. $D_\varphi \cot \varphi = -(1 + \cot^2 \varphi)$

226. $D_\varphi \sec \varphi = \sec \varphi \tan \varphi$

227. $D_\varphi \csc \varphi = - \csc \varphi \cot \varphi$

228. $D_\varphi \text{ver} \sin \varphi = \sin \varphi$

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229. $D_x \text{arc} \sin x = \frac{1}{\sqrt{1 - x^2}}$

230. $D_x \text{arc} \cos x = \frac{-1}{\sqrt{1 - x^2}}$

231. $D_x \text{arc} \tan x = \frac{1}{1 + x^2}$

232. $D_x \text{arc} \cot x = \frac{-1}{1 + x^2}$

233. $D_x \text{arc} \sec x = \frac{1}{x\sqrt{x^2 - 1}}$

234. $$D_x \text{ arc csc } x = \frac{-1}{x\sqrt{x^2 - 1}}$$

235. $$D_x \text{ arc ver sin } x = \frac{1}{\sqrt{2x - x^2}}$$

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236. $$D_\phi \log \sin \varphi = \cot \varphi$$

237. $$D_\varphi \log \cos \varphi = -\tan \varphi$$

238. $$D_\varphi \log \tan \varphi = \frac{2}{\sin 2\varphi}$$

239. $$D_\varphi \log \csc \varphi = -\cot \varphi$$

240. $$D_\varphi \log \sec \varphi = \tan \varphi$$

241. $$D_\varphi \log \cot \varphi = \frac{-2}{\sin 2\varphi}$$

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242. $$D_\varphi \log \tan (45^\circ \pm \varphi) = \frac{\pm 2}{\cos 2\varphi}$$

243. $$D_x\, e^{ax} = a\, e^{ax}$$

244. $$D_u \log u = \frac{1}{u}$$

VIII. Series.

$\mathfrak{S}$ = Naperian base of logarithms.

245. $\mathfrak{S} = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \dots$

246. $\mathfrak{S}^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \dots$

247. $f(x+h) = f(x) + \frac{h}{1} Df(x) + \frac{h^2}{1.2} D^2 f(x) + \dots$

248. $f(x+h) = \mathfrak{S}^{hD} f(x)$

249. $\sin\varphi = \varphi - \frac{\varphi^3}{1.2.3} + \frac{\varphi^5}{1.2.3.4\ 5} - \dots$

250. $\cos\varphi = 1 - \frac{\varphi^2}{1.2} + \frac{\varphi^4}{1.2.3.4} - \dots$

251. $\tan\varphi = \varphi + \frac{1}{3}\varphi^3 + \frac{2}{15}\varphi^5 + \frac{17}{215}\varphi^7 + \dots$

252. $\text{arc} \sin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \dots$

253. $\text{arc} \cos x = \frac{1}{2}\pi - \text{arc} \sin x$

$= (1 - x) + \frac{1}{2}\frac{(1-x^3)}{3} + \frac{1.3}{2.4}\frac{(1-x^5)}{5} + \dots$

254. $\text{arc} \tan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$

255. $\frac{1}{i}\sin\varphi i = \varphi + \frac{\varphi^3}{1.2.3} + \frac{\varphi^5}{1.2.3.4.5} + \dots$

256. $\cos\varphi i = 1 + \frac{\varphi^2}{1.2} + \frac{\varphi^4}{1.2.3.4} + \dots$

IX. Miscellaneous Formulas.

To reduce Arc to Angle and the reverse.

Let φ°, φ', φ'' express the number of degrees, minutes, or seconds in an angle or arc.

Let arc φ°, arc φ', arc φ'' express the linear magnitude of arcs corresponding to the angles φ°, φ', and φ''; the unit of linear magnitude being the radius.

Let R°, R', R'' express the radius in degrees, minutes, or seconds; that is, the number of degrees, minutes, or seconds in an arc which is equal to the radius in length.

φ used alone shall express, as usual, the linear magnitude of an arc, radius being unity.

257. $\varphi = \varphi^\circ \times \text{arc } 1^\circ = \varphi' \times \text{arc } 1' = \varphi'' \times \text{arc } 1''$

258. $$\varphi = \frac{\varphi^\circ}{R^\circ} = \frac{\varphi'}{R'} = \frac{\varphi''}{R''}$$

259. $$\varphi^\circ = R^\circ \times \varphi = \frac{\varphi}{\text{arc } 1^\circ}$$

260. $$\varphi' = R' \times \varphi = \frac{\varphi}{\text{arc } 1'}$$

261. $$\varphi'' = R'' \times \varphi = \frac{\varphi}{\text{arc } 1''}$$

262. $\begin{cases} \text{arc } 1^\circ = 0.01745\ 32925\ 19943 \\ \text{arc } 1' = 0.00029\ 08882\ 08666 \\ \text{arc } 1'' = 0.00000\ 48481\ 36811 \end{cases}$

263. $\begin{cases} \log \text{arc } 1^\circ = \bar{2}.24187\ 73675\ 90827 \\ \log \text{arc } 1' = \bar{4}.46372\ 61172\ 07184 \\ \log \text{arc } 1'' = \bar{6}.68557\ 48668\ 235405 \end{cases}$

264. $\begin{cases} R^\circ = 57^\circ.29577\ 95131 \\ R' = 3437'.7467\ 707849 \\ R'' = 206264''.806247 \end{cases}$

265. $\begin{cases} \log R^\circ = 1.75812\ 26324\ 09172 \\ \log R' = 3.53627\ 38827\ 92816 \\ \log R'' = 5.31442\ 51331\ 76459 \end{cases}$

266. $\begin{cases} \sin 1' = 0.00029\ 08882\ 04564 \\ \sin 1'' = 0.00000\ 48481\ 36809 \end{cases}$

267. $\begin{cases} \pi = 3.14159\ 26535\ 89793\ 23846 \\ \text{Common } \log \pi = 0.49714\ 98726\ 94134 \\ \text{Naperian } \log \pi = 1.14472\ 98858\ 49400 \end{cases}$

Accurate Computation of Small Arcs.

[See page 24.]

268. $\log \sin \varphi = \log \varphi'' + 4.685\ 5749 - \frac{1}{3}(10 - \log \cos \varphi)$

269. $\log \tan \varphi = \log \varphi'' + 4.685\ 5749 + \frac{2}{3}(10 - \log \cos \varphi)$

270. $\log \varphi'' = \log \sin \varphi + 5.314\ 4251 + \frac{1}{3}(10 - \log \cos \varphi) - 10$

271. $\log \varphi'' = \log \tan \varphi + 5.314\ 4251 - \frac{2}{3}(10 - \log \cos \varphi) - 10$

Logarithms.

Let M_a denote the modulus of a system of logarithms of which the base is a, and $\log_a x$ the logarithm of x in that system. Express the Naperian logarithm of x by $\mathrm{l}\, x$.

272. $$M_a = \frac{\log_a x}{\mathrm{l}\, x} \qquad \log_a x = M_a \,\mathrm{l}\, x$$

273. $$M_a = \log_a \epsilon$$

274. $$\begin{aligned} M_{10} &= \text{Modulus of Common Logarithms} \\ &= \text{Common Logarithm of } \epsilon \\ &= 0.43429\ 44819 \end{aligned}$$

275. $$\frac{1}{M_{10}} = 2.30258\ 50930$$

276. Common logarithm of $M_{10} = \bar{1}.63778\ 43113$

277. $$\epsilon = 2.718281828459$$

278. $$\mathrm{l}\,(x+h) = \mathrm{l}\, x + \frac{h}{x} - \frac{h^2}{2\,x^2} + \frac{h^3}{3\,x^3} - \ldots$$
in which $h < x$.

279. $$\mathrm{l}\,(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots$$
in which x may be real or imaginary provided its modulus be less than unity.

Differential Variations of the parts of a Plane Triangle.

280. $$\begin{cases} da = \cos C\ db + \cos B\ dc + b \sin C\ dA \\ db = \cos A\ dc + \cos C\ da + c \sin A\ dB \\ dc = \cos B\ da + \cos A\ db + a \sin B\ dC \end{cases}$$

Differential Variations of the parts of a Spherical Triangle.

281. $$\begin{cases} da = \cos C\ db + \cos B\ dc + \sin b \sin C\ dA \\ db = \cos A\ dc + \cos C\ da + \sin c \sin A\ dB \\ dc = \cos B\ da + \cos A\ db + \sin a \sin B\ dC \end{cases}$$

The Potential Functions.

282. $$\mathrm{p} \sin \varphi = \frac{1}{\mathrm{i}} \sin \varphi\, \mathrm{i} = \tfrac{1}{2} (\mathfrak{e}^{\varphi} - \mathfrak{e}^{-\varphi})$$

283. $$\mathrm{p} \cos \varphi = \cos \varphi\, \mathrm{i} = \tfrac{1}{2} (\mathfrak{e}^{\varphi} + \mathfrak{e}^{-\varphi})$$

284. $$\mathrm{p} \tan \varphi = \frac{\mathrm{p} \sin \varphi}{\mathrm{p} \cos \varphi} = \frac{1}{\mathrm{i}} \tan \varphi\, \mathrm{i} = \frac{\mathfrak{e}^{2\varphi} - 1}{\mathfrak{e}^{2\varphi} + 1}$$

285. $$\mathrm{p} \cos^2 \varphi - \mathrm{p} \sin^2 \varphi = 1$$

286. $$1 - \mathrm{p} \tan^2 \varphi = \mathrm{p} \sec^2 \varphi$$

287. $$\mathrm{p} \sin (\alpha \pm \beta) = \mathrm{p} \sin \alpha\ \mathrm{p} \cos \beta \pm \mathrm{p} \cos \alpha\ \mathrm{p} \sin \beta$$

288. $$\mathrm{p} \cos (\alpha \pm \beta) = \mathrm{p} \cos \alpha\ \mathrm{p} \cos \beta \pm \mathrm{p} \sin \alpha\ \mathrm{p} \sin \beta$$

The definitions (**282**–**284**) and the fundamental equations (**285**–**288**) give rise to a system of formulas analogous to the formulas of common Trigonometry. The following results are useful: —

289. $$D_\varphi \text{ p sin } \varphi = \text{p cos } \varphi$$

290. $$D_\varphi \text{ p cos } \varphi = \text{p sin } \varphi$$

291. $$D_\varphi \text{ p tan } \varphi = 1 - \text{p tan}^2 \varphi = \text{p sec}^2 \varphi$$

292. $$D_x \text{ arc p sin } x = \frac{1}{\sqrt{1+x^2}}$$

293. $$D_x \text{ arc p cos } x = \frac{1}{\sqrt{x^2-1}}.$$

294. $$D_x \text{ arc p tan } x = \frac{1}{1-x^2}$$

THE END.

Cambridge: Electrotyped and Printed by Welch, Bigelow, & Co.